Exercise 15E: Solutions

1 First find the inverse of $\mathbf{A}$.

$$\det(\mathbf{A}) = 3 \times -1 - -1 \times 4 = 1$$

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$$

a If $\mathbf{AX} = \mathbf{K}$ then $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K}$

$$\therefore \mathbf{IX} = \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \times -1 + 1 \times 2 \\ -4 \times -1 + 3 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

b If $\mathbf{AX} = \mathbf{K}$ then $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K}$

$$\therefore \mathbf{IX} = \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \times -2 + 1 \times 3 \\ -4 \times -2 + 3 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 17 \end{bmatrix}$$

2 a $\begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

$$\text{Determinant} = -2 \times 1 - 4 \times 3 = -14$$

$$\text{Inverse} = \frac{1}{-14}\begin{bmatrix} 1 & -4 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{14} \times 6 + \frac{2}{7} \times 1 \\ \frac{3}{14} \times 6 + \frac{1}{7} \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{7} \\ \frac{10}{7} \end{bmatrix}$$

$$x = -\frac{1}{7},\ y = \frac{10}{7}$$

b $\begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\text{Determinant} = -1 \times 4 - 2 \times -1 = -2$$

$$\text{Inverse} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \times -1 + 1 \times 2 \\ -\frac{1}{2} \times -1 + \frac{1}{2} \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ \frac{3}{2} \end{bmatrix}$$

$$x = 4,\ y = \frac{3}{2} \text{ or } 1.5$$

3 Solve the simultaneous equations

$$2x - 3y = 7$$

$$3x + y = 5$$

$$\begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\text{Determinant} = 2 \times 1 - -3 \times 3 = 11$$

$$\text{Inverse} = \frac{1}{11}\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 1 \times 7 + 3 \times 5 \\ -3 \times 7 + 2 \times 5 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 22 \\ -11 \end{bmatrix}$$

$$x = 2,\ y = -1$$

The point of intersection is $(2,\ -1)$.

4 If x is the number of books they are buying and y is the number of CDs they are buying, then the following equations apply.

$$4x + 4y = 120$$

$$5x + 3y = 114$$

$$\begin{bmatrix} 4 & 4 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 120 \\ 114 \end{bmatrix}$$

$$\text{Determinant} = 4 \times 3 - 4 \times 5 = -8$$

$$\text{Inverse} = \frac{1}{-8}\begin{bmatrix} 3 & -4 \\ -5 & 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3 & 4 \\ 5 & -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3 & 4 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 120 \\ 114 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -3 \times 120 + 4 \times 114 \\ 5 \times 120 + -4 \times 114 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 96 \\ 144 \end{bmatrix}$$

$$x = 12,\ y = 18$$

One book costs \$12, a CD costs \$18.

5 **a** $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

b $\det(\mathbf{A}) = 2 \times -6 - -3 \times 4 = 0$, so the matrix is non-invertible.

c Yes. For example $x = 0,\ y = -1$ is an obvious solution.

d You should notice that the second equation is simply the first with both sides multiplied by 2.
There is an infinite number of solutions to these equations, just as there is an infinite number of ordered pairs that make $2x - 3y = 3$ a true equation.

6 **a** $\mathbf{A}^{-1}\mathbf{C}$

b $\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{C}$

c $\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$

d $\mathbf{A}^{-1}\mathbf{C} - \mathbf{B}$

e $\mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

f $(\mathbf{A} - \mathbf{B})\mathbf{A}^{-1} = \mathbf{I} - \mathbf{B}\mathbf{A}^{-1}$